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On Colocalization of the Category of Finite Groups

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Abstract

In this paper we define a colocalization functor with respect to a torsion theory which is defined in the category of finite groups and prove the uniqueness of it.

0 Introduction

In [2] the author defined a torsion theory for the category of finite groups. From Examples 2.1 and 2.2 in [2] it is natural to consider colocalization of a group. It is possible to define a colocalization with respect to a torsion theory in the category of finite groups like in abelian categories. But it is almost impossible to prove the uniqueness of colocalization. In abelian categories it is easy to to check the uniqueness of colocalization. The author almost gave up to consider colocalization in the category of finite groups. But why is the uniqueness of colocalization necessary? The uniqueness is necessary only to get a colocalization functor! So at first it is not necessary to define a colocalization of each group. We only need to define a colocalization functor and to prove the uniqueness of it up to natural equivalence. After that a colocalization of each group can be defined by means of a colocalization functor.

Anyway the author succeeded to define a colocalization functor and to prove the uniqueness of it. The remaining problem is to prove the existence of the colocalization functors of Examples of 2.1 and 2.2 in [2]. They seem fairly difficult. So far the author does not have any idea.

1 $\mathcal{F}_t$-projective groups

$\mathcal{G}$ denotes the category of finite groups. For terminologies of torsion theories we refer to [2]. Throughout this paper let t be an idempotent radical in $\mathcal{G}$ and $(\mathcal{T}_t, \mathcal{F}_t)$ the associated torsion theory. Moreover suppose there exists a right exact endofunctor of $\mathcal{G}$ (we denote it by c) with a natural transformation $\phi : c \to t$ such that $c(X) \in \mathcal{T}_t$, $Im\,\phi_X = t(X)$ and $Ker\,\phi_X \in \mathcal{F}_t$ for all $X \in \mathcal{G}$. We check the properties of c.

Proposition 1.1 *t is epi-preserving.*

Proof *Suppose $X \to X'' \to 1$ is exact. Then since c is right exact c is also epi-preserving.*

Hence we have the following commutative diagram with exact upper raw and columns.

$$\begin{tikzcd} c(X) \arrow[r] \arrow[d] & c(X) \arrow[r] \arrow[d] & 1 \\ t(X) \arrow[r] \arrow[d] & t(X'') \arrow[d] & \\ 1 & 1 & \end{tikzcd}$$

Thus t is epi-preserving.

In [2, Proposition 1.4] we proved that t is epi-preserving iff $\mathcal{F}_t$ is closed under epimorphic images.

Definition 1 *A torsion theory $(\mathcal{T}, \mathcal{F})$ in $\mathcal{G}$ is said to be cohereditary if $\mathcal{F}$ is closed under epimorphic images.*

Thus in the present paper, $(\mathcal{T}_t, \mathcal{F}_t)$ is a cohereditary torsion theory.

Definition 2 *$X \in \mathcal{G}$ is $\mathcal{F}_t$-projective (or t-projective) if $Hom(X, -)$ is exact on all short exact sequences $1 \to Y' \to Y \to Y'' \to 1$ with $Y' \in \mathcal{F}_t$.*

Lemma 1.2 $c(X) = 1$ *iff* $X \in \mathcal{F}_t$.

Proof *If $c(X) = 1$ then $t(X) = 1$ since $Im\,\phi_X = t(X)$. Hence $X \in \mathcal{F}_t$. Conversely suppose $X \in \mathcal{F}_t$. Then $1 \to Ker\,\phi_X \to c(X) \to t(X) \to 1$ is exact. But $t(X) = 1$ implies $Ker\,\phi_X = c(X)$. Therefore $c(X) \in \mathcal{T}_t \cap \mathcal{F}_t = \{1\}$.*

Theorem 1.3 *$c^2(X)$ is $\mathcal{F}_t$-projective for all $X \in \mathcal{G}$.*

Proof *Suppose the diagram*

$$\begin{tikzcd} & & & c^2(X) \arrow[d, "f"] & \\ 1 \arrow[r] & Y' \arrow[r] & Y \arrow[r] & Y'' \arrow[r] & 1 \end{tikzcd}$$

with $Y' \in \mathcal{F}_t$ is given. ϕ_X induces an exact sequence $1 \to Ker\,\phi_X \to c(X) \to t(X) \to 1$. Hence $c^2(X) \simeq ct(X)$. Thus $c(c^2(X)) \simeq c^3(X) \simeq c^2(c(X)) \simeq ct(c(X)) \simeq c^2(X)$ since $t(c(X)) = c(X)$. This implies that $c(c^2(X))$ and $c^2(X)$ have the same order. Moreover ϕ_X is an epimorphism for all $X \in \mathcal{T}_t$ since $Im\,\phi_X = t(X)$. In particular $\phi_{c^2(X)}$ is an epimorphism. But $c(c^2(X))$ and $c^2(X)$ have the same order. Therefore $\phi_{c^2(X)}$ is an isomorphism. Hence we have the following commutative diagram.

$$\begin{tikzcd} & & & c(c^2(X)) \arrow[d, "c(f)"] & c^2(X) \arrow[l, "{\phi_{c^2(X)}}^{-1}"'] \arrow[ddl, "f"] & \\ & & c(Y) \arrow[d, "\phi_Y"] & c(Y'') \arrow[l, phantom, "\simeq"] \arrow[d, "\phi_{Y''}"] & & \\ 1 \arrow[r] & Y' \arrow[r] & Y \arrow[r] & Y'' \arrow[r] & 1 & \end{tikzcd}$$

Now it is clear that $c^2(X)$ is $\mathcal{F}_t$-projective.

By the method of the proof of this theorem we get the following.

Corollary 1.4 *If $\phi_X : c(X) \to X$ is an isomorphism, then X is $\mathcal{F}_t$-projective.*

Theorem 1.5 *$c^2 \simeq c \cdot t$, c^2 is idempotent and c^2 is right exact.*

Proof *Let $1 \to X' \to X \to X'' \to 1$ be exact. We have the following commutative diagram with an exact upper raw.*

$$\begin{array}{ccccccc} c(X') & \to & c(X) & \to & c(X'') & \to & 1 \\ \downarrow & & \downarrow & & \downarrow & & \\ t(X') & \to & t(X) & \to & t(X'') & & \end{array}$$

Put $I = Im\,(c(X') \to c(X))$ and $K = Ker\,(c(X') \to c(X))$. Then since $K \subset Ker\,\phi_{X'}$, $c(K) = 1$ holds. Hence $c^2(X') \simeq c(I)$ from the exact sequence $1 \to K \to c(X') \to I \to 1$. On the other hand, since $1 \to I \to c(X) \to c(X'') \to 1$ is exact, $c(I) \to c^2(X) \to c^2(X'') \to 1$ is exact. Therefore c^2 is right exact. By the way, $c^2(X) \simeq c \cdot t(X)$ from the exact sequence $1 \to Ker\,\phi_X \to c(X) \to t(X) \to 1$. Therefore $c \cdot t$ is right exact. Finally we show that c^2 is idempotent. It is enough to show that ct is idempotent. $(ct)^2 = c(tc)t \simeq c^2t \simeq ct^2 = ct$. This completes the proof.

Remark Since $Im\,\phi_X = t(X)$, ϕ induces the natural transformation $\phi' : c \to t$. With this the following diagram is commutative from the fact that $\phi : c \to 1_{\mathcal{G}}$ is a natural transformation.

$$\begin{array}{ccc} c^2(X) & \xrightarrow{\phi_{c(X)}} & c(X) \\ \downarrow c(\phi'_X) & & \downarrow \phi'_X \\ c(t(X)) & \xrightarrow{\phi_{t(X)}} & t(X) \end{array}$$

Note that $c(\phi'_X)$ is an isomorphism. Let $i : t \to 1_{\mathcal{G}}$ be the canonical natural transformation (i.e., $i_X : t(X) \to X$ is the inclusion map). Let $\mu = i \cdot \phi' \cdot c(\phi) : c^2 \to 1_{\mathcal{G}}$. In other words, $\mu_X = c^2(X) \xrightarrow{\phi_{c(X)}} c(X) \xrightarrow{\phi'_X} t(X) \xrightarrow{i_X} X$. Then c^2 is idempotent and right exact, $Ker\,\mu_X \in \mathcal{F}_t$, $Im\,\mu_X = t(X)$ and $c^2(X)$ is $\mathcal{F}_t$-projective. We shall see that c^2 is the colocalization functor.

2 Colocalization functor and the uniqueness of it

Let c be an endofunctor of $\mathcal{G}$ with a natural transformation $\phi : c \to 1_{\mathcal{G}}$. Suppose the following conditions hold.

(1) c is right exact.

(2) c is idempotent, i.e., there exists a natural equivalence $c^2 \simeq c$.

(3) For each $X \in \mathcal{G}$,

  (i) $Ker\,\phi_X \in \mathcal{F}_t$.

  (ii) $Im\,\phi_X = t(X)$.

Then we call $\phi_X : c(X) \to X$ the colocalization of X with respect to $(\mathcal{T}_t, \mathcal{F}_t)$, and c the colocalization functor. Of course $c(X)$ is $\mathcal{F}_t$-projective. Moreover $\phi_{c(-)} : c^2 \to c$ is a natural equivalence.

In the preceding section, c^2 is a colocalization functor by the remark of that section.

Theorem 2.1 *Let c_1, c_2 be two colocalization functors with respect to $(\mathcal{T}_t, \mathcal{F}_t)$ with natural transformations $\phi : c_1 \to 1_{\mathcal{G}}$ and $\psi : c_2 \to 1_{\mathcal{G}}$. Then there exists a natural equivalence $\mu : c_1 \simeq c_2$ such that the diagram*

$$\begin{tikzcd} c_1(X) \arrow[r, "\phi_X"] \arrow[d, "\mu_X"'] & X \\ c_2(X) \arrow[ur, "\psi_X"'] & \end{tikzcd}$$

is commutative for each $X \in \mathcal{G}$.

Proof ϕ_X *and* ψ_X *induce canonical homomorphisms* $\phi'_X : c_(X) \to t(X)$ *and* $\psi'_X : c_2(X) \to t(X)$ *respectively. In the proof of theorem 1.3, let* $1 \to Y' \to Y \to Y'' \to 1$ *be the exact sequence* $1 \to Ker\,\psi_X \to c_2(X) \overset{\psi'_X}{\to} t(X) \to 1$ *and* $f = \phi'_{c(X)} : c_1(X) \to t(X)$. *Then* $\mu_x = \phi_{c_2(X)} \cdot c_1(\psi'_X)^{-1} \cdot c_1(\phi'_X) \cdot \phi_{c_1(X)}{}^{-1} : c_1(X) \to c_2(X)$ *is an epimorphism. Similarly there exists an epimorphism* $c_2(X) \to c_1(X)$. *Thus* μ_X *is an isomorphism. In this case* $\mu : c_1 \to c_2$ *is a natural equivalence. since*

$$\begin{tikzcd} c_1(X) \arrow[dr, "\phi'_X"] \arrow[dd, "\mu_X"'] & \\ & t(X) \\ c_2(X) \arrow[ur, "\psi'_X"'] & \end{tikzcd}$$

is commutative, μ *is the desired natural equivalence.*

References

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[3] H. Tachikawa and K. Ohtake, *Colocalization and Localization in Abelian Categories*, J. Algebra, 56-1 (1979), 1-23.